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Langevin Equation with Multi-Poissonian Noise

Akira Onuki[1]

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A general master equation is shown to be equivalent to a Langevin equation whose noise is expressed as a linear superposition of Poissonian random variables (multi-Poissonian noise). As typical examples, a birth and death process and a Boltzmann–Langevin equation are given.

KEY WORDS: Poissonian noise; Langevin equation; master equation; Boltzmann–Langevin equation.

1. INTRODUCTION

A Langevin equation describing the macroscopic behavior of a system can be derived from a microscopic equation using the projection operation technique.[1–3] However, it is difficult to determine microscopically the stochastic nature of a random source term in a Langevin equation. It has been assumed on the basis of phenomenological arguments to be a Gaussian Markov stochastic process in most applications to physical problems. It should be stressed that it may be another stochastic process.[4] We give in this paper such an example, which is clear and easy but has not been discussed explicitly before.

2. MULTI-POISSONIAN PROCESS

A stochastic differential equation can be written generally in the form[5]

$$\frac{\partial}{\partial t} X(t) = F(X(t), t; \omega) \tag{1}$$

Here, ω represents an event or state which is realized following some given distribution on the set of events or states Ω. This may be considered as a

[1] Department of Physics, Kyushu University, Fukuoka, Japan.

generalization of a Langevin equation. A Markovian process can be characterized by the condition that the infinitesimal increment

$$\Delta X(t) = \int_t^{t+\Delta t} dt'\, F(X(t'), t'; \omega) \tag{2}$$

has a distribution dependent only on the instantaneous value of $X(t)$. That is, the distribution of $\Delta X(t)$ does not depend on the past details of the process, namely, it does not depend on $X(s)$, where s is prior to t. In this case we can construct an evolution equation for the distribution function of $X(t)$. For example, if $\Delta X(t)$ is Gaussian, the distribution of $X(t)$ is governed by a Fokker–Planck equation. We consider in this paper the case in which $\Delta X(t)$ is expressed as a linear superposition of Poissonian random variables. Such a stochastic process will be called Markovian and multi-Poissonian. We shall show that the distribution of $X(t)$ with multi-Poissonian noise obeys a master equation and vice versa. The infinitesimal increment $\Delta X(t)$, Eq. (2), will be assumed to be of the form

$$\Delta X(t) = \sum_i c_i\, \Delta J_i(X(t)) \tag{3}$$

where the $\Delta J_i(X(t))$ are independent Poissonian random variables taking nonnegative integral values and have the mean values

$$\langle \Delta J_i(X(t)); X\rangle = \Delta t W_i(X) + O(\Delta t^2) \tag{4}$$

Here, $\langle \cdots; X\rangle$ is the conditional average with the value of $X(t)$ fixed as X. The positive quantity $W(X)$ may depend on X. That is, the probability of finding $\Delta J_i(X(t))$ to be n with the condition $X(t) = X$ is given by

$$\frac{1}{n!}\,[\Delta t W_j(X)]^n e^{-\Delta t W_j(X)} \tag{5}$$

so that the characteristic function of $\Delta J_i(X(t))$ is given by

$$\begin{aligned}\langle \exp[i\xi\, \Delta J_i(X(t))]; X\rangle &= \exp\{[\exp(i\xi) - 1]\, \Delta t W_i(X)\} \\ &= 1 + [\exp(i\xi) - 1]\, \Delta t W_i(X) + O(\Delta t^2)\end{aligned} \tag{6}$$

where Δt is considered as a positive, infinitesimal number. The nth cumulant of $\Delta J_i(X(t))$ coincides with the first to the first order in Δt for arbitrary n. It should be noted here that a linear superposition of independent Poissonian random variables is called compound Poissonian in the literature (see appendix). Using Eq. (6), we can derive an equation for the distribution of $X(t)$, $P(X, t)$: we first note the equation

$$\begin{aligned}\frac{\partial}{\partial t} P(X, t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\,(P(X, t + \Delta t) - P(X, t)) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \frac{d\xi}{2\pi}\, e^{-i\xi X} \langle (e^{i\xi \Delta X(t)} - 1) e^{i\xi X(t)} \rangle\end{aligned} \tag{7}$$

where $\langle \cdots \rangle$ is the average over all possible events. Next from Eqs. (3)–(6) we obtain

$$\langle \exp[i\xi\, \Delta X(t)] - 1; X \rangle = \prod_i \exp\{[\exp(i\xi c_i) - 1)\, \Delta t W_i(X)\} - 1$$
$$= \Delta t \sum_i [\exp(i\xi c_i) - 1] W_i(X) + O(\Delta t^2) \qquad (8)$$

where we have used the fact that the characteristic function of the sum of independent random variables is the product of their characteristic functions. Thus we obtain the master equation

$$\frac{\partial}{\partial t} P(X, t) = \sum_i \left[\exp\left(-c_i \frac{\partial}{\partial X} \right) - 1 \right] W_i(X) P(X, t) \qquad (9)$$

We remark here that any sample function $X(t, \omega)$ changes discontinuously in time by an amount $c_i n$ each time because the $\Delta J_i(X(t))$ take nonnegative integral values only. Therefore, the time derivative in Eq. (1) should be interpreted as $(1/\Delta t)\, \Delta X(t)$ where the limit of infinitesimal Δt is not taken. This is assumed implicitly even for the case of Gaussian noise, where any sample function is continuous but not differentiable almost everywhere.

3. RANDOM FORCE

Next, in order to see the relation between the differential form (1) and the incremental form (3), we rewrite Eq. (3) in the following Langevin form:

$$\frac{\partial}{\partial t} X(t) = \sum_i c_i W_i(X(t)) + R(t) \qquad (10)$$

where $W_i(X)$ are defined in Eq. (4). The average of $R(t)$ vanishes and its distribution is stochastically independent of the past process $X(s)$, where $s < t$. The source term $R(t)$ will be called "a random force." Mathematically, the quantity $R(t)$ consists of the δ-functions of time corresponding to the discontinuous change in $X(t)$, so that it is convenient to consider the increment of $R(t)$ written as

$$\Delta R(t) = \int_t^{t+\Delta t} dt'\, R(t') = \Delta X(t) - \sum_i c_i W_i(X(t))\, \Delta t \qquad (11)$$

Then the characteristic function of the random increment $\Delta R(t)$ becomes

$$\langle e^{i\xi \Delta R(t)}; X \rangle = 1 + \Delta t \sum_i (e^{i\xi c_i} - i\xi c_i - 1) W_i(X) + O(\Delta t^2) \qquad (12)$$

The cumulants of $\Delta R(t)$ are all first order in Δt, while in the case of Gaussian noise only the second cumulant is of order Δt. The time correlation function of the random force $R(t)$ should be defined by

$$\langle R(t_1)R(t_2)\rangle = \lim_{\Delta t_1 \to 0} \lim_{\Delta t_2 \to 0} \frac{1}{\Delta t_1 \Delta t_2} \langle \Delta R(t_1)\, \Delta R(t_2)\rangle \tag{13}$$

where $\Delta R(t_1)$ is the increment in the time interval $[t_1, t_1 + \Delta t_1]$ and $\Delta R(t_2)$ that in $[t_2, t_2 + \Delta t_2]$. Equation (13) is nonvanishing only when the two time intervals overlap. Let Δt be the overlapping time; then we obtain

$$\langle \Delta R(t_1)\, \Delta R(t_2)\rangle = \Delta t \sum_i c_i \langle W_i(X(t))\rangle + O(\Delta t^2) \tag{14}$$

Using the mathematical identity

$$\lim_{\Delta t_1 \to 0} \lim_{\Delta t_2 \to 0} \frac{1}{\Delta t_1 \Delta t_2} \Delta t = \delta(t_1 - t_2) \tag{15}$$

we find

$$\langle R(t_1)R(t_2)\rangle = \delta(t_1 - t_2) \sum_i c_i \langle W_i(X(t))\rangle \tag{16}$$

The correlation time of the random force $R(t)$ is infinitesimally small. This is because all impacts $\Delta J_i(X(t))$ in Eq. (3) cause instantaneous changes in $X(t)$, independent of each other.

4. BIRTH AND DEATH PROCESS

In the following we give two examples of a multi-Poissonian Markov process. For the first example, let $X(t)$ be a population in the system considered with $X(t)$ changing by integral values when birth or death events take place. The incremental change $\Delta X(t)$ is assumed to be multi-Poissonian:

$$\Delta X(t) = \sum_{r = \pm 1, \pm 2, \ldots} r\, \Delta J_r(X(t)) \tag{17}$$

with

$$\langle \Delta J_r(X); X\rangle = \Delta t\, W(X \to X + r) + O(\Delta t^2) \tag{18}$$

where $\Delta J_r(X)$ is the number of birth and death events in which $X(t)$ changes by r. The distribution of $X(t)$ obviously obeys the following familiar master equation

$$\frac{\partial}{\partial t} P(X, t) = \sum_r [W(X - r \to X)P(X - r, t) - W(X \to X + r)P(X, t)] \tag{19}$$

We note that when $X(t)$ is extensive, that is, proportional to the system size

Ω, the normalized random force $R(t)/\Omega$ becomes Gaussian to leading order Ω^{-1} [$\exp(i\xi c_i)$ in Eq. (12) can be expanded in c_i].[6]

5. BOLTZMANN–LANGEVIN EQUATION

As the second example, we consider the fluctuating motion of the one-body distribution $F(\mathbf{r}, \mathbf{v}, t)$, the average motion of which obeys the Boltzmann equation.[7–9] Let us consider mutually independent particles undergoing collisions with randomly distributed scatterers. Their distribution in μ space, $F(\mathbf{r}, \mathbf{v}, t)$, obeys the following Boltzmann–Langevin equation:

$$\frac{\partial}{\partial t} F(\mathbf{r}, \mathbf{v}, t) + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} F(\mathbf{r}, \mathbf{v}, t)$$

$$= \frac{1}{\Delta t}\int d\mathbf{r}'\, d\mathbf{v}'\, [-\Delta J(\mathbf{r}, \mathbf{v} \to \mathbf{r}'\mathbf{v}'; t) + \Delta J(\mathbf{r}', \mathbf{v}' \to \mathbf{r}, \mathbf{v}; t)] \tag{20}$$

Here, $\Delta J(\mathbf{r}, \mathbf{v} \to \mathbf{r}', \mathbf{v}'; t)$ is the number of collisional events taking place in the time interval $[t, t + \Delta t]$ in which the colliding particle is located at $(\mathbf{r}, \mathbf{v})$ before the collision and is transferred to $(\mathbf{r}', \mathbf{v}')$ after the collision. This number is assumed to be Poissonian with the mean value

$$\langle \Delta J(\mathbf{r}, \mathbf{v} \to \mathbf{r}', \mathbf{v}'; t); F(t)\rangle = \Delta t\, \delta(\mathbf{r} - \mathbf{r}') n_i W(\mathbf{v} \to \mathbf{v}') F(\mathbf{r}, \mathbf{v}, t) \tag{21}$$

where n_i is the impurity density and $W(\mathbf{v} \to \mathbf{v}')$ is the transition probability associated with the collision. We note that it is possible to derive Eq. (21) microscopically in the limit of small collision duration time.[10] It is convenient to introduce the characteristic functional

$$Q(\{\eta\}, t) = \left\langle \exp\left[i\int d\mathbf{r}\, d\mathbf{v}\, \eta(\mathbf{r}, \mathbf{v}) F(\mathbf{r}, \mathbf{v}, t)\right]\right\rangle \tag{22}$$

which obeys

$$\frac{\partial}{\partial t} Q + \int d\mathbf{r}\, d\mathbf{v}\, \eta(\mathbf{r}, \mathbf{v})\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} \frac{\delta}{\delta\eta(\mathbf{r}\cdot\mathbf{v})} Q$$

$$= \int d\mathbf{r}\, d\mathbf{v} \int d\mathbf{v}'\, n_i W(\mathbf{v} \to \mathbf{v}')\{\exp[i\eta(\mathbf{r}, \mathbf{v}') - i\eta(\mathbf{r}, \mathbf{v})] - 1\}$$

$$\times \frac{1}{i}\frac{\delta}{\delta\eta(\mathbf{r}, \mathbf{v})} Q \tag{23}$$

This can be solved to give

$$Q(\{\eta\}, t) = \exp\left(\int \{\exp[i\eta(\mathbf{r}, \mathbf{v})] - 1\}\langle F(\mathbf{r}, \mathbf{v}, t)\rangle\, d\mathbf{r}\, d\mathbf{v}\right) \tag{24}$$

If we divide μ space into cells, then (24) clearly indicates that the particle numbers in cells are independently Poissonian, each with the mean value $\Delta N = \langle F(\mathbf{r}, \mathbf{v}, t)\rangle\, \Delta\mathbf{r}\, \Delta\mathbf{v}$. Some authors have considered the particle number distributions to be Gaussian.[9,11] This is allowed in the following two cases. (i) One is the case in which small-angle collisions are so dominant that it is allowable to use the following expansion with respect to the velocity change:

$$\begin{aligned}\exp[i\eta(\mathbf{r}, \mathbf{v}') &- i\eta(\mathbf{r}, \mathbf{v})] - 1 \\ &= i(\mathbf{v}' - \mathbf{v})\cdot\frac{\partial}{\partial\mathbf{v}}\,\eta(\mathbf{r}, \mathbf{v}) \\ &\quad - \frac{1}{2}(\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}'):\frac{\partial}{\partial\mathbf{v}}\frac{\partial}{\partial\mathbf{v}}\,\eta(\mathbf{r}, \mathbf{v}) - \frac{1}{2}\left[(\mathbf{v}' - \mathbf{v})\cdot\frac{\partial}{\partial\mathbf{v}}\,\eta(\mathbf{r}, \mathbf{v})\right]^2 + \cdots\end{aligned} \tag{25}$$

If we retain only the first three terms in Eq. (25), we readily find that Eq. (23) reduces to a Fokker–Planck equation, showing that $F(\mathbf{r}, \mathbf{v}, t)$ obeys a Gaussian Markov stochastic process. We can mention as an example electrons coupled with acoustic phonons. (ii) The second is the case in which the average numbers ΔN are much greater than unity, since a Poisson distribution becomes Gaussian in the limit of a large mean value. It is important to note here that there is an arbitrariness in dividing μ space into cells, so that ΔN may be made much greater than unity by choosing an appropriate cell size. Papers based on the Gaussian assumption are thus justified for either of the above two cases.

A Boltzmann–Langevin equation for a system of interacting particles takes the following, slightly different form:

$$\Delta F(1, t) + \mathbf{v}_1\cdot\frac{\partial}{\partial\mathbf{r}_1} F(1, t)\,\Delta t = \int d1'\, d2\, d2'\, [\Delta J^{12}_{1'2'}(t) - \Delta J^{1'2'}_{12}(t)] \tag{26}$$

where $1 = (\mathbf{r}_1, \mathbf{v}_1)$,... represent points in μ space and $\Delta J^{1'2'}_{12}(t)$ is the number of collisions in which two particles are located at 1 and 2 before the collision and at $1'$ and $2'$ after the collision. This number $\Delta J^{1'2'}_{12}(t)$ is Poissonian with the mean value

$$\begin{aligned}\langle \Delta J^{1'2'}_{12}(t); F(t)\rangle = \delta(\mathbf{r}_1 - \mathbf{r}_2)\,\delta(\mathbf{r}_1 - \mathbf{r}_1')\,\delta(\mathbf{r}_1 - \mathbf{r}_2') \\ \times W(\mathbf{v}_1, \mathbf{v}_2 \to \mathbf{v}_1', \mathbf{v}_2')F(1, t)F(2, t)\end{aligned} \tag{27}$$

where $W(\mathbf{v}_1, \mathbf{v}_2 \to \mathbf{v}_1', \mathbf{v}_2')$ is the transition probability associated with the binary collision. It should be noted that Eq. (24) is not satisfied when particles interact with one another; in this case, $\log Q$ may be expanded in powers of $e^{i\eta} - 1$ as

$$\begin{aligned}\log Q(\{\eta\}, t) = \int d\mathbf{r}\, d\mathbf{v}\, \{\exp[i\eta(\mathbf{r}, \mathbf{v})] - 1\}\langle F(\mathbf{r}, \mathbf{v}, t)\rangle \\ + \frac{1}{2!}\int d\mathbf{r}\, d\mathbf{v} \int d\mathbf{r}'\, d\mathbf{v}'\, \{\exp[i\eta(\mathbf{r}, \mathbf{v})] - 1\} \\ \times \{\exp[i\eta(\mathbf{r}', \mathbf{v}')] - 1\} g(\mathbf{r}, \mathbf{v}, \mathbf{r}', \mathbf{v}', t) + \cdots\end{aligned} \tag{28}$$

where $g(\mathbf{r}, \mathbf{v}, \mathbf{r}', \mathbf{v}', t)$,... are the cumulant correlation functions.[12] If we neglect them, the equation for the average $\langle F(\mathbf{r}, \mathbf{v}, t)\rangle$ reduces to the usual Boltzmann equation. Equations for the cumulant correlation functions form an infinite hierarchy of equations when there are collisions between particles. The cumulant correlation functions defined by Eq. (28) cannot predict short-range correlations (those in the region $|\mathbf{r}_i - \mathbf{r}_j| \lesssim$ force range), because Boltzmann–Langevin equations such as Eq. (26) describe the motion of long-wavelength components of the particle density in μ space (that is, short-wavelength components are eliminated by coarse-graining).[11] However, a Boltzmann–Langevin equation for gaseous systems with interactions between particles can predict long-range correlations extending much farther than the force range in nonequilibrium states. The present author has examined the long-range pair correlation in the presence of a steady laminar flow by constructing the equation for g.[10]

APPENDIX

The increment $\Delta X(t)$, Eq. (3), can be considered to consist of instantaneous changes caused by independent impacts such as bombardment of physical particles. The number of such impacts has the Poisson distribution with the mean value

$$\Delta N = \Delta t \sum_i W_i(X) \tag{A1}$$

and the distribution of the change x caused by one impact is common to all impacts and is given by

$$P(x; X) = \frac{1}{\sum_i W_i(X)} \sum_i \delta(x - c_i) W_i(X) \tag{A2}$$

Denoting the characteristic function of $P(x; X)$ by $f(\xi; X)$, we obtain the characteristic function of the total change in the form

$$\begin{aligned}\langle \exp[i\xi\, \Delta X(t)]; X\rangle &= \sum_n \frac{(\Delta N)^n}{n!} [\exp(-\Delta N)] f(\xi; X)^n \\ &= \exp[(f(\xi; X) - 1)\, \Delta N] \end{aligned} \tag{A3}$$

which is Eq. (8). Here, $f(\xi; X)^n$ is the characteristic function of the sum of n independent changes. In the literature the distribution whose characteristic function takes the form (A3) is called compound Poissonian, where $f(\xi)$ may be a characteristic function of any distribution, and the stochastic process $X(t)$ with W_i independent of X is called a compound Poisson process.[13] That is, a multi-Poissonian random variable, which is expressed as a linear superposition of independent Poissonian random variables, is compound

Poissonian. However, we have allowed $W_i(X)$ to depend on X, so that the characteristic function of $\Delta X(t)$, Eq. (8), coincides with that of the compound Poissonian distribution only to the first order in Δt.

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